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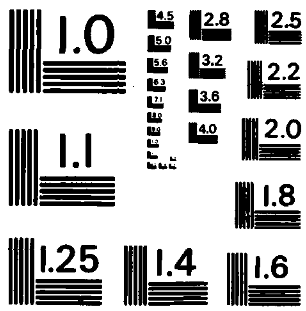

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LIKELIHOOD RATIO TESTS FOR RELATIONSHIPS BETWEEN TWO COVARIANCE MATRICES

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ABSTRACT

Likelihood ratio tests for hypotheses on relationships between two population covariance matrices Σ_1 and Σ_2 are derived on the basis of the sample covariance matrices having Wishart distributions. The specific hypotheses considered are (i) $\Sigma_2 = \sigma^2\Sigma_1$, (ii) $\Sigma_2 = \Gamma + \sigma^2\Sigma_1$, (iii) $\Sigma_2 = \Gamma + \Sigma_1$ where Γ may be n.n.d. or arbitrary and the rank of Γ is less than that of Σ_1. Some applications of these tests are given.

sigma

gamma

AMS Subject Classification: 62H15, 62H25

KEY WORDS: Analysis of dispersion, familial correlations, likelihood ratio tests, MANOVA, Principal components, Wilks Λ.

LIKELIHOOD RATIO TESTS FOR RELATIONSHIPS
BETWEEN TWO COVARIANCE MATRICES

C. Radhakrishna Rao

1. INTRODUCTION

Let $\underset{\sim}{S}_1$: p×p and $\underset{\sim}{S}_2$: p×p be two random symmetric matrices having Wishart distributions $W_p(n_1,\underset{\sim}{\Sigma}_1)$ and $W_p(n_2,\underset{\sim}{\Sigma}_2)$ respectively, where n_1 and n_2 are degrees of freedom, and $\underset{\sim}{\Sigma}_1$ and $\underset{\sim}{\Sigma}_2$ are population covariance matrices. In this paper the likelihood ratio tests are derived for the following hypotheses on $\underset{\sim}{\Sigma}_1$ and $\underset{\sim}{\Sigma}_2$:

H_1: $\underset{\sim}{\Sigma}_2 = \sigma^2\underset{\sim}{\Sigma}_1$, σ^2 unknown,

H_2: $\underset{\sim}{\Sigma}_2 = \underset{\sim}{\Gamma} + \sigma^2\underset{\sim}{\Sigma}_1$, $\underset{\sim}{\Gamma}$ is n.n.d. and $\rho(\underset{\sim}{\Gamma}) = k < p$, σ^2 unknown,

H_3: $\underset{\sim}{\Sigma}_2 = \underset{\sim}{\Gamma} + \underset{\sim}{\Sigma}_1$, $\underset{\sim}{\Gamma}$ is n.n.d. and $\rho(\underset{\sim}{\Gamma}) = k < p$,

H_4: $\underset{\sim}{\Sigma}_2 = \underset{\sim}{\Gamma} + \underset{\sim}{\Sigma}_1$, $\rho(\underset{\sim}{\Gamma}) = k < p$,

where $\rho(\underset{\sim}{A})$ = the rank of the matrix $\underset{\sim}{A}$ and n.n.d. stands for non-negative definiteness.

Applications of the above tests to problems of inference on "familial correlations" introduced by the author (see Rao, 1945 and the follow up in Rao, 1953) are discussed.

The following well known results and notations are used.

(i) If $\underset{\sim}{\Sigma}_1$ and $\underset{\sim}{\Sigma}_2$ are nonsingular, then $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$ are nonsingular with probability 1

(ii) If $\underset{\sim}{S}_1$ is nonsingular, then there exist matrices $\underset{\sim}{P}$ and $\underset{\sim}{T} = (\underset{\sim}{P}')^{-1}$ such that

$$S_1 = T T', \quad S_2 = T \Lambda T' \tag{1.1}$$

$$P' S_1 P = I, \quad P' S_2 P = \Lambda \tag{1.2}$$

where Λ is the diagonal matrix with the roots $\ell_1, \ldots, \ell_p$ of $|S_2 - \lambda S_1| = 0$ as the diagonal elements (see Rao, 1973, p. 41).

(iii) Let A: $p \times p$ be a real symmetric matrix and T_i: $p \times k_i$, $i = 1, \ldots, r$ be such that

$$\Sigma k_i = p, \quad \rho(T_1 : \ldots : T_r) = p, \tag{1.3}$$

$$T_i' A T_j = 0, \quad T_i' T_j = 0, \quad i \neq j. \tag{1.4}$$

Then there exists a choice of eigen vectors $R_1, \ldots, R_p$, of A such that the columns of each T_i depend on an exclusive subset of $R_1, \ldots, R_p$.

(iv) If X: $p \times p$, then the matrix derivative with respect to $X = (X_{ij})$ of a scalar function $f(X)$ is defined by

$$\frac{\partial f}{\partial X} = \left(\frac{\partial f}{\partial X_{ij}}\right): p \times p. \tag{1.5}$$

For particular choices of f, we have (Rao, 1973, p.72),

$$\frac{\partial |X|}{\partial X} = |X| (X^{-1})' \tag{1.6}$$

$$\frac{\partial \operatorname{tr} M X}{\partial X} = M' \tag{1.7}$$

$$\frac{\partial \operatorname{tr} X^{-1} M}{\partial X} = -(X^{-1} M X^{-1})' \tag{1.8}$$

(v) The log likelihood of Σ_1, Σ_2 given S_1, S_2 (considering only the terms depending on Σ_1, Σ_2) multiplied by 2 is

$$L(\Sigma_1, \Sigma_2 | S_1, S_2)$$

$$= -n_1 \log |\Sigma_1| - \operatorname{tr} \Sigma_1^{-1} S_1 - n_2 \log |\Sigma_2| - \operatorname{tr} \Sigma_2^{-1} S_2 \qquad (1.9)$$

so that

$$\left[\frac{\partial L}{\partial \Sigma_1}\right]' = -n_1 \Sigma_1^{-1} + \Sigma_1^{-1} S_1 \Sigma_1^{-1}, \quad \left[\frac{\partial L}{\partial \Sigma_2}\right]' = -n_2 \Sigma_2^{-1} + S_2 \Sigma_2^{-1}. \qquad (1.10)$$

[Note that in taking the derivatives we do not consider Σ_1 and Σ_2 as symmetric matrices. This does not matter so long as the optimum solutions for Σ_1 and Σ_2 turn out to be symmetric.]

2. TEST FOR $\Sigma_2 = \sigma^2 \Sigma_1$ (σ^2 unknown)

Substituting $\sigma^2 \Sigma_1$ for Σ_2 in $L(\Sigma_1, \Sigma_2 | S_1, S_2)$ of (1.9), and taking derivatives with respect of Σ_1 and σ^2 using the formulae (1.6)-(1.8), we have

$$\left[\frac{\partial L}{\partial \Sigma_1}\right]' = -(n_1+n_2)\Sigma_1^{-1} + \Sigma_1^{-1}(S_1 + \sigma^{-2} S_2)\Sigma_1^{-1} = 0 \qquad (2.1)$$

$$\frac{\partial L}{\partial \sigma^2} = -pn_2 + \sigma^{-2} \operatorname{tr} \Sigma_1^{-1} S_2 = 0. \qquad (2.2)$$

From (2.1),

$$(n_1+n_2)\Sigma_1 = S_1 + \sigma^{-2} S_2 = T(I + \sigma^{-2}\Lambda)T'$$

$$(n_1+n_2)^{-1}\Sigma_1^{-1} = P(I + \sigma^{-2}\Lambda)^{-1}P' \qquad (2.3)$$

where T and P are as defined in (1.1) and (1.2). Eliminating Σ_1^{-1} from (2.2) using (2.3), we obtain the equation for estimating the unknown σ^2 as

$$\frac{pn_2}{n_1+n_2} = \sum_{i=1}^{p} \frac{\ell_i}{\ell_i+\sigma^2} = \sum_{i=1}^{p} \frac{n_2 m_i}{n_2 m_i + n_1 \sigma^2} \qquad (2.4)$$

where $\ell_1,\ldots,\ell_p$ are the roots of $|S_2 - \lambda S_1| = 0$ and $m_i = n_1\ell_i/n_2$. The equation (2.4) has only one non-negative solution which we represent by $\hat{\sigma}^2$. Then the estimate of Σ_1 is

$$\hat{\Sigma}_1 = \frac{S_1+\hat{\sigma}^{-2}S_2}{n_1+n_2}. \tag{2.5}$$

The likelihood ratio test (LRT) for testing H_1: $\Sigma_2 = \sigma^2\Sigma_1$ is based on the difference

$$\sup_{\Sigma_1,\Sigma_2} L(\Sigma_1,\Sigma_2|S_1,S_2) - \sup_{\Sigma_1,\sigma^2} L(\Sigma_1,\sigma^2\Sigma_1|S_1,S_2)$$

$$= \log \prod_{i=1}^{p} \left[\frac{1}{m_i^{n_2}\,\hat{\sigma}^{2n_1}} \left(\frac{n_2m_i+n_1\hat{\sigma}^2}{n_2+n_1} \right)^{n_1+n_2} \right]. \tag{2.6}$$

The statistic (2.6) has an asymptotic χ^2 distribution on $[p^2+p-2)/2]$ degrees of freedom (d.f.) when n_1 and n_2 tend to infinity.

It may be recalled that the LR test for $\Sigma_2 = \Sigma_1$ (Kshirsagar, 1978, p.404) is

$$\log \pi \left[\left(\frac{n_1+n_2m_i}{n_1+n_2} \right)^{n_1+n_2} \frac{1}{m_i^{n_2}} \right] \tag{2.7}$$

which can be written as the sum of (2.6) and

$$\log \pi \left[\left(\frac{n_1+n_2m_i}{n_1\hat{\sigma}^2+n_2m_i} \right)^{n_1+n_2} \hat{\sigma}^{2n_1} \right]. \tag{2.8}$$

The statistic (2.8) has χ^2 distribution on 1 d.f. when $\Sigma_2 = \sigma^2\Sigma_1$ and n_1,n_2 are large, and can be used to test the hypothesis $\sigma^2 = 1$.

Further, if a confidence interval for σ^2 is needed, we can use the statistic

$$\log \pi \left[\left(\frac{n_1\sigma^2+n_2m_i}{n_1\hat{\sigma}^2+n_2m_i} \right)^{n_1+n_2} \left(\frac{\hat{\sigma}}{\sigma} \right)^{2n_1} \right] \tag{2.9}$$

as χ^2 on 1 d.f.

A hypothesis of the type $\Sigma_2 = \sigma^2\Sigma_1$ occurs in examining whether two response vectors differ by a scaler multiplier. For example, in the evaluation of drugs, two drugs will be considered equivalent if their response vectors, x, y, differ by a scalar multiplier, since by a change of dosage the effects may be made equal. Such a hypothesis specifies that $E(x) = c\,E(y)$ and $D(x) = c^2\,D(y)$. We have considered only the hypothesis $D(x) = c^2\,D(y)$. Testing of the hypothesis, $E(x) = c\,E(y)$, under the condition $D(x) = D(y)$ was considered by Cochran (1943) and Kraft, Olkin and van Eeeden (1972).

3. TEST FOR $\Sigma_2 = \Gamma + \sigma^2\Sigma_1$

Under the hypothesis $\Sigma_2 = \Gamma + \sigma^2\Sigma_1$ with Γ as an n.n.d. matrix of rank $k < p$ and σ^2 unknown, we can write

$$\Sigma_1 = R_1R_1' + \ldots + R_pR_p' = RR' \tag{3.1}$$

$$\Sigma_2 = \lambda_1R_1R_1' + \ldots + \lambda_kR_kR_k' + \sigma^2(R_{k+1}R_{k+1}' + \ldots + R_pR_p') = R\,\Delta\,R' \tag{3.2}$$

where Δ is a diagonal matrix with $\lambda_1 > \ldots > \lambda_k > \sigma^2, \ldots, \sigma^2$ as diagonal elements. We shall maximize

$$L(\Sigma_1, \Sigma_2 | S_1, S_2) + \mathrm{tr}\, M_1(\Sigma_1 - RR') + \mathrm{tr}\, M_2(\Sigma_2 - R\,\Delta\,R')$$

where M_1 and M_2 are matrices of Lagrangian multipliers. The optimizing equations are

$$-n_1\Sigma_1^{-1} + \Sigma_1^{-1}S_1\Sigma_1^{-1} + M_1 = 0, \quad -n_2\Sigma_2^{-1} + \Sigma_2^{-1}S_2\Sigma_2^{-1} + M_2 = 0 \tag{3.3}$$

$$M_2R\,\Delta + M_1R = 0 \tag{3.4}$$

$$R_i'M_2R_i = 0,\ i=1,\dots,k,\ \sum_{i=k+1}^{p} R_i'M_2R_i = 0 \tag{3.5}$$

With $U' = R^{-1}$, we have from (3.3)

$$n_1 I = U'S_1U + R'M_1R, \tag{3.6}$$

$$n_2 I = \Delta^{-1}U'S_2U + R'M_2R\Delta. \tag{3.7}$$

Adding (3.6) and (3.7) and using (3.4), we have

$$(n_1+n_2)I = U'S_1U + \Delta^{-1}U'S_2U. \tag{3.8}$$

The equations (3.6), (3.7) and (3.8) $\Rightarrow$

$$U_i'S_1U_i = n_1,\ U_i'S_2U_i = n_2\lambda_i,\ U_i'S_1U_j = 0 = U_i'S_2U_j,\ i \neq j \tag{3.9}$$
$$i,j = 1,\dots,k$$

$$U_i'S_1V = 0,\ U_i'S_2V = 0,\ i=1,\dots,k,\ V = (U_{k+1}:\dots:U_p) \tag{3.10}$$

The results (3.9) and (3.10) show, by using the results (1.3) and (1.4), that the estimates of λ_i, U_i and V are

$$\hat{\lambda}_i = m_i,\ \hat{U}_i = n_1^{1/2}P_i,\ i=1,\dots,k,\ \hat{V} = (P_{k+1}:\dots:P_p)G \tag{3.11}$$

where G: $(p-k)\times(p-k)$ is any matrix such that

$$G'(I + \sigma^{-2}K)G = (n_1+n_2)\ I \tag{3.12}$$

In (3.12), K is the diagonal matrix with $\ell_{k+1},\dots,\ell_p$, the last (p-k) eigen values of $|S_2 - \lambda S_1| = 0$ as the diagonal elements. The equation (3.12) together with (3.5)-(3.7) provide the estimate $\hat{\sigma}^2$ of σ^2 as the nonnegative root of the equation

$$\frac{(p-k)n_2}{n_1+n_2} = \sum_{i=k+1}^{p} \frac{n_2 m_i}{n_2 m_i + n_1 \sigma^2} . \tag{3.13}$$

The LRT of the hypothesis H_2: $\Sigma_2 = \Gamma + \sigma^2 \Sigma_1$ is based on the difference

$$\sup_{\Sigma_1, \Sigma_2} L(\Sigma_1, \Sigma_2 | S_1, S_2) - \sup_{H_2} L(\Sigma_1, \Sigma_2 | S_1, S_2)$$

$$= \log \prod_{i=k+1}^{p} \left[\left(\frac{n_2 m_i + n_1 \hat{\sigma}^2}{n_1+n_2} \right)^{n_1+n_2} \frac{1}{m_i^{n_2} \, \hat{\sigma}^{2n_1}} \right] \tag{3.14}$$

which has a χ^2 distribution, asymptotically as n_1 and $n_2 \to \infty$, on $[(p-k)(p-k+1)-2]/2$ d.f.

Muirhead (1978) obtained a representation of the conditional asymptotic distribution of $\ell_{k+1},\ldots,\ell_p$ given $\ell_1,\ldots,\ell_k$ and the last p-k roots are equal, and observed that by neglecting a linkage factor, this distribution is the same as that of the roots of $V_2 V_1^{-1}$ where V_1 and V_2 have Wishart distributions $W_{p-k}(n_1, B)$ and $W_{p-k}(n_2-k, \sigma^2 B)$ respectively. In such a case we may expect a better approximation to the χ^2 distribution by considering the statistic (3.14) with n_2 replaced by n_2-k.

4. TEST FOR $\Sigma_2 = \Gamma + \Sigma_1$ (Γ, n.n.d.)

Under the hypothesis $\Sigma_2 = \Gamma + \Sigma_1$ with Γ as an n.n.d. matrix of rank k, we can write

$$\Sigma_1 = R_1 R_1' + \ldots + R_p R_p' \tag{4.1}$$

$$\Sigma_2 = \lambda_1 R_1 R_1' + \ldots + \lambda_k R_k R_k' + R_{k+1} R_{k+1}' + \ldots + R_p R_p' \tag{4.2}$$

where

$$\lambda_1 > \ldots > \lambda_k > 1. \tag{4.3}$$

In order to compute the likelihood ratio test, it is necessary to obtain the estimates of $\lambda_1, \ldots, \lambda_k$ subject to the condition (4.3) which is somewhat difficult.

However, we can approach the problem of testing the hypothesis $\underset{\sim}{\Sigma}_2 = \underset{\sim}{\Gamma} + \underset{\sim}{\Sigma}_1$ by breaking up into two parts. One is for testing the hypothesis, $\underset{\sim}{\Sigma}_2 = \underset{\sim}{\Gamma} + \sigma^2\underset{\sim}{\Sigma}_1$ (i.e., the last p-k eigen values of $\underset{\sim}{\Sigma}_2$ with respect to $\underset{\sim}{\Sigma}_1$ are equal), and another for testing the hypothesis, $\sigma^2 = 1$ given that the first hypothesis holds.

The appropriate test statistic for the hypothesis, $\underset{\sim}{\Sigma}_2 = \underset{\sim}{\Gamma} + \sigma^2\underset{\sim}{\Sigma}_1$, is given in (3.14). If this hypothesis is not disproved, we proceed to test the hypothesis $\sigma^2 = 1$ by using the statistic

$$z = \left[\frac{n_1 n_2 (p-k)}{2(n_1+n_2)}\right]^{1/2} (\hat{\sigma}^2 - 1) \tag{4.4}$$

which is asymptotically distributed as a normal deviate. To deduce the result (4.5), observe that $\hat{\sigma}^2$ is a root of the equation

$$\frac{(p-k)n_2}{n_1+n_2} = \sum_{i=k+1}^{p} \frac{n_2 m_i}{n_2 m_i + n_1 \sigma^2} \tag{4.5}$$

so that by the δ-method

$$\delta\,\hat{\sigma}^2 = \frac{1}{p-k}\,\Sigma\,\delta m_i \tag{4.6}$$

under the assumption that the true values of $m_{k+1}, \ldots, m_p$ are all equal to σ^2.

Thus the asymptotic distribution of $\hat{\sigma}^2$ is the same as that of the average $(m_{k+1}+\ldots+m_p)/(p-k)$. Then, using the results on the asymptotic distribution of the functions of the roots $m_{k+1},\ldots,m_p$ (see Fang and Krishnaiah, 1982 and Muirhead, 1978), the asymptotic distribution of (4.4) is seen to be normal.

As observed earlier, it is difficult to derive the LR test of the hypothesis that the last (p-k) roots of $\underset{\sim}{\Sigma}_2\underset{\sim}{\Sigma}_1^{-1}$ are equal with common value unity. Even if the exact LR test is obtained, it may not have an asymptotic χ^2 distribution. [It may be noted that in a similar context, of the principal component analysis, the statistic for testing that the last (p-k) roots of a covariance matrix are all equal to a given value, given by Anderson (1963) and quoted by Kshirsagar (1978, p. 448) is not an LRT.]

However, substituting $\hat{\sigma}^2 = 1$ in (3.14) we obtain the statistic

$$\log \prod_{i=k+1}^{p} \left[\left(\frac{n_1+n_2m_i}{n_1+n_2}\right)^{n_1+n_2} \frac{1}{m_i^{n_2}}\right] \tag{4.7}$$

which provides an overall measure of the difference between the vectors $(m_{k+1},\ldots,m_p)$ and $(1,\ldots,1)$. The statistic (4.7) can be written as the sum of (3.14) and

$$\log \prod_{i=k+1}^{p} \left[\left(\frac{n_1+n_2m_i}{n_1\hat{\sigma}^2+n_2m_i}\right)^{n_1+n_2} \hat{\sigma}^{2n_1}\right]. \tag{4.8}$$

If the true values of $m_{k+1},\ldots,m_p$ are equal, then the statistic (4.8) is asymptotically equivalent to

$$\frac{n_1n_2(p-k)}{2(n_1+n_2)}(\hat{\sigma}^2-1)^2 \tag{4.9}$$

which is the square of the statistic (4.4), and hence is asymptotically distributed as χ^2 on 1 d.f. Thus (4.8) is an alternative statistic to (4.4)

to test the hypothesis that the common value of the last (p-k) roots is unity.

In view of the remark made at the end of Section 3 based on Muirhead's observation, the statistic (4.7) is asymptotically distributed as χ^2 on $(p-k)(p-k+1)/2$ d.f. if the last (p-k) roots of $\Sigma_2\Sigma_1^{-1}$ are equal with the common value unity. But as observed earlier, it is more meaningful to use the statistics (3.14) and (4.8) [or (4.4)] by breaking the hypothesis into two parts, one specifying the equality of the roots and other specifying the common value.

If the common value specified is c, instead of unity, the statistic (4.8) is changed to

$$\log \prod_{i=k+1}^{p} \left[\left(\frac{n_1 c + n_2 m_i}{n_1 \hat{\sigma}^2 + n_2 m_i} \right)^{n_1+n_2} \left(\frac{\hat{\sigma}^2}{c} \right)^{n_1} \right]. \tag{4.10}$$

Also, there may be some theoretical advantage in replacing n_2 by (n_2-k) in defining the statistics (3.14), (4.8) and (4.10).

5. TEST FOR $\Sigma_2 = \Gamma + \Sigma_1$

In Section 4, we considered the hypothesis $\Sigma_2 = \Gamma + \Sigma_1$ where Γ is n.n.d. and $\rho(\Gamma) = k$. If the n.n.d. condition is not imposed, then the hypothesis $\Sigma_2 = \Gamma + \Sigma_1$ implies that <u>some</u> (p-k) eigen values of Σ_2 with respect to Σ_1 are equal to unity. In such a case Σ_1 and Σ_2 can be written as

$$\Sigma_1 = R_1R_1' + \ldots + R_pR_p' \tag{5.1}$$

$$\Sigma_2 = \lambda_1R_1R_1' + \ldots + \lambda_kR_kR_k' + R_{k+1}R_{k+1}' + \ldots + R_pR_p' \tag{5.2}$$

where λ_i need not be greater than unity.

The equations for estimating the unknown parameters under the hypothesis $\Sigma_2 = \Gamma + \Sigma_1$ are

$$-n_1\,\underset{\sim}{\Sigma}_1^{-1}+\underset{\sim}{\Sigma}_1^{-1}\underset{\sim}{S}_1\underset{\sim}{\Sigma}_1^{-1}+\underset{\sim}{M}_1=0,\; -n_2\,\underset{\sim}{\Sigma}_2^{-1}+\underset{\sim}{\Sigma}_2^{-1}\underset{\sim}{S}_2\underset{\sim}{\Sigma}_2^{-1}+\underset{\sim}{M}_2=0 \tag{5.3}$$

$$\underset{\sim}{M}_2\underset{\sim}{R}\,\underset{\sim}{\Delta}+\underset{\sim}{M}_1\underset{\sim}{R}=0,\; \underset{\sim}{R}_i'\underset{\sim}{M}_2\underset{\sim}{R}_i=0,\; i=1,\ldots,k \tag{5.4}$$

where $\underset{\sim}{\Delta}$ is a diagonal matrix with $\lambda_1,\ldots,\lambda_k$, $1,\ldots,1$ as diagonal elements. The equations (5.3) and (5.4) are the same as those in (3.3)-(3.5) except for the equation $\sum_{i=k+1}^{p} \underset{\sim}{R}_i'\underset{\sim}{M}_2\underset{\sim}{R}_i = 0$ corresponding to σ^2.

Proceeding as in Section 3, the LRT for the hypothesis $\underset{\sim}{\Sigma}_2=\underset{\sim}{\Gamma}+\underset{\sim}{\Sigma}_1$ is seen to be

$$\inf_j \log \prod_{i=j+1}^{j+(p-k)} \left[\left(\frac{n_2 m_i + n_1}{n_2+n_1}\right)^{n_1+n_2} \frac{1}{m_i^{\,n_2}}\right] \tag{5.5}$$

which is asymptotically distributed as χ^2 on $(p-k)(p-k+1)/2$ d.f. The statistic (5.5) is different from (4.7).

6. FAMILIAL CORRELATIONS

In an early paper (Rao, 1945), the author introduced the concept of "familial correlations" as a generalization of the intraclass correlation. They arose in a natural way in defining a single measure of correlation between members (such as brothers) of a family with respect to a number of measurements. Typically we have a b×p matrix variable

$$\underset{\sim}{X}=\begin{pmatrix} x_{11},\ldots,x_{p1} \\ . \quad \ldots \quad . \\ x_{1b},\ldots,x_{pb} \end{pmatrix} = \begin{pmatrix} \underset{\sim}{x}_1' \\ \vdots \\ \underset{\sim}{x}_b' \end{pmatrix} \tag{6.1}$$

where the i-th row vector $\underset{\sim}{x}_i'$ corresponds to the measurements of p characteristics on the i-th member of a family. For instance, if we are considering brothers in a family, the row may correspond to the parity of a brother. In such a case, a natural model for the means and variances and covariances of the variables in (6.1) is

$$E(\underset{\sim}{x}_i) = \underset{\sim}{\mu}_i, \quad i = 1,\dots,k \tag{6.2}$$

$$D(\underset{\sim}{X}) = \begin{pmatrix} \underset{\sim}{A} & \underset{\sim}{B} & \dots & \underset{\sim}{B} \\ \underset{\sim}{B} & \underset{\sim}{A} & \dots & \underset{\sim}{B} \\ \vdots & \vdots & \dots & \vdots \\ \underset{\sim}{B} & \underset{\sim}{B} & \dots & \underset{\sim}{A} \end{pmatrix} . \tag{6.3}$$

Krishnaiah and Lee (1974) and Olkin (1973) considered the problem of testing the structure of the dispersion matrix of X as given in (6.3).

In earlier papers of the author (Rao, 1945 and the follow up in Rao, 1953), the mean vectors $\underset{\sim}{\mu}_i$ were taken to be the same (which is valid when the members of a family are not distinguishable) and $D(\underset{\sim}{X})$ is as in (6.3). Under this model, familial correlations were defined as intraclass correlations for suitably chosen linear functions of the p measurements.

If we have observations on $\underset{\sim}{X}$ from N families, then we can write down the Analysis of Dispersion (MANOVA) for a two way classification (families × parity) in the usual way.

Table 1. Analysis of dispersion for two way classification by family and parity

Due to	D.F.	Sums of squares and products (SSP)	Mean squares and products (MSP)	E(MSP)
Families	N-1	$(F_{ij}) = \underset{\sim}{F}$	(f_{ij})	$b\underset{\sim}{B} + \underset{\sim}{A} - \underset{\sim}{B} = \underset{\sim}{\Gamma} + \underset{\sim}{\Sigma}_1$
Parity	b-1	$(P_{ij}) = \underset{\sim}{P}$	(p_{ij})	$\underset{\sim}{\Phi} + \underset{\sim}{A} - \underset{\sim}{B} = \underset{\sim}{\Phi} + \underset{\sim}{\Sigma}_1$
Interaction	(N-1)(b-1)	$(W_{ij}) = \underset{\sim}{W}$	(W_{ij})	$\underset{\sim}{A} - \underset{\sim}{B} = \underset{\sim}{\Sigma}_1$

In Table 1, Φ represents the non-centrality parameter which becomes a null matrix if $\underset{\sim}{\mu}_i$ are all equal, and

$$\underset{\sim}{F} \sim W_p(N-1,\ \underset{\sim}{\Gamma}+\underset{\sim}{\Sigma}_1),\ \underset{\sim}{W} \sim W_p((N-1)(k-1),\ \underset{\sim}{\Sigma}_1) \tag{6.4}$$

$$\underset{\sim}{P} \sim W_p(N-1,\ \underset{\sim}{\Phi},\underset{\sim}{\Sigma}_1) \tag{6.5}$$

are all independently distributed. The joint distribution of the familial correlations can be obtained from that of the roots of the equation $|\underset{\sim}{F} - \lambda\underset{\sim}{W}| = 0$, derived by Roy (1939).

Two hypotheses of interest in such studies are

$$H_{01}:\ \underset{\sim}{\mu}_1 = \ldots = \underset{\sim}{\mu}_b \quad \text{or} \quad \underset{\sim}{\Phi} = 0 \tag{6.6}$$

$$H_{02}:\ \rho(\underset{\sim}{B}) = \rho(\underset{\sim}{\Gamma}) = k. \tag{6.7}$$

The hypothesis H_{01} can be tested by Wilks $\underset{\sim}{\Lambda} = |\underset{\sim}{W}|/|\underset{\sim}{P}+\underset{\sim}{W}|$, and the hypothesis H_{02} can be tested by using the statistic (5.5), writing $\underset{\sim}{F}$ for $\underset{\sim}{S}_2$ and $\underset{\sim}{W}$ for $\underset{\sim}{S}_1$.

The familial correlations, $\rho_1,\rho_2,\ldots$, defined by Rao (1945, 1953) are the roots of the equation $|\underset{\sim}{B} - \rho\underset{\sim}{A}| = 0$. The maximum root is the maximum correlation between any two members of a family with respect to a linear combination of the measurements. The number of nonzero familial correlations is equal to the rank of $\underset{\sim}{B}$, and the hypothesis (6.7) is therefore relevant in drawing inferences on familial correlations.

It is seen that if $\ell_1,\ell_2,\ldots$ are the roots of the equation $|\underset{\sim}{F} - \lambda\underset{\sim}{W}| = 0$, then the estimate r_i of ρ_i is obtained from the relationship

$$\frac{(1+\overline{b-1}\ r_i)}{(b-1)(1-r_i)} = \ell_i,\ i = 1,2,\ldots. \tag{6.8}$$

If $\underset{\sim}{\mu}_1 = \ldots = \underset{\sim}{\mu}_b$, then the estimates of $\rho_1,\rho_2,\ldots$ are obtained from the roots $\ell_1',\ell_2',\ldots$ of $|\underset{\sim}{F} - \lambda(\underset{\sim}{W}+\underset{\sim}{P})| = 0$ by the formula

$$\frac{(n-1)}{n(b-1)}\ \frac{1+\overline{b-1}\ r_i}{1-r_i} = \ell_i' \tag{6.9}$$

which shows the relevance of the hypothesis (6.6).

In the statement of the hypothesis (6.7), no further condition was imposed on $\underset{\sim}{B}$. If $\underset{\sim}{B}$ is not n.n.d., then some of the familial correlations will be negative.

Let the matrix variable $\underset{\sim}{X}$: b×p have a structure of the type

$$X_{ij} = \mu_{ij} + \gamma_j + \epsilon_{ij} \tag{6.10}$$

where μ_{ij} are constants, and γ_j and ϵ_{ij} are stochastic variables representing family effects (common to all members of a family) and random effects respectively, such that

$$\begin{aligned} &\mathrm{Cov}(\gamma_j,\epsilon_{ij}) = 0,\ \mathrm{Cov}(\epsilon_{ij},\epsilon_{km}) = 0,\ i \neq k \\ &D(\gamma_1,\ldots,\gamma_p) = \underset{\sim}{B},\ D(\epsilon_{i1},\ldots,\epsilon_{ip}) = \underset{\sim}{C},\ i = 1,\ldots,b. \end{aligned} \tag{6.11}$$

Then

$$D(\underset{\sim}{X}) = \begin{pmatrix} \underset{\sim}{A} & \underset{\sim}{B} & \ldots & \underset{\sim}{B} \\ \underset{\sim}{B} & \underset{\sim}{A} & \ldots & \underset{\sim}{B} \\ \cdot & \cdot & \ldots & \cdot \\ \underset{\sim}{B} & \underset{\sim}{B} & \ldots & \underset{\sim}{A} \end{pmatrix} \tag{6.12}$$

where $\underset{\sim}{A} = \underset{\sim}{B} + \underset{\sim}{C}$ and $\underset{\sim}{B}$ is n.n.d. The rank of $\underset{\sim}{B}$ is equal to the number of linearly independent variables among $\gamma_1,\ldots,\gamma_p$ (family effects specific to the p measurements). Thus, if the covariance matrix of $\underset{\sim}{X}$ is specified to be of the form (6.12), then the hypothesis (6.10) on the structure of the random variable $\underset{\sim}{X}$ with k linearly independent γ_i is equivalent to

$$H_{03}:\ \underset{\sim}{B} \text{ is n.n.d. and } \rho(\underset{\sim}{B}) = k.$$

Such a hypothesis can be tested by using the statistics (3.14) and (4.5), with $\underset{\sim}{F} = \underset{\sim}{S}_2$, $\underset{\sim}{W} = \underset{\sim}{S}_1$ and the corresponding changes in the degrees of freedom.

There is some similarity between the hypotheses considered in the present paper with those of Fisher (1939) and Anderson (1951). Let $\underset{\sim}{\gamma}_1,\ldots,\underset{\sim}{\gamma}_N$ be

the unknown vectors of family effects in N observed families. The problems considered by Fisher and Anderson relate to hypotheses on $\gamma_1, \ldots, \gamma_N$ considered as fixed parameters. In the present paper, we consider γ_i as stochastic and test hypotheses concerning the common covariance matrix of γ_i.

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Likelihood ratio tests for hypotheses on relationships between two population covariance matrices Σ_1 and Σ_2 are derived on the basis of the sample covariance matrices having Wishart distributions. The specific hypotheses considered are (i) $\Sigma_2 = \sigma^2\Sigma_1$, (ii) $\Sigma_2 = \Gamma + \sigma^2\Sigma_1$, (iii) $\Sigma_2 = \Gamma + \Sigma_1$ where Γ may be n.n.d. or arbitrary and the rank of Γ is less than that of Σ_1. Some applications of these tests are given.

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